

MATHEMAGIC ACTIVITY BOOK

CLASS -XII

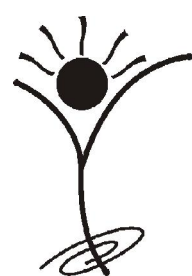
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• LEARNING FOR LIFE •

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SYLLABUS GUIDELINES

CLASS - XII

Based on CBSE & ICSE Syllabus

Compulsory for all

RELATIONS AND FUNCTIONS

1. Relations and Functions:

Types of relations: reflexive, symmetric, transitive and equivalence relations. One to one and onto functions, composite functions, inverse of a function. Binary operations.

2. Inverse Trigonometric Functions:

Definition, range, domain, principal value branches. Graphs of inverse trigonometric functions. Elementary properties of inverse trigonometric functions.

ALGEBRA

1. Matrices:

Concept, notation, order, equality, types of matrices, zero matrix, transpose of a matrix, symmetric and skew symmetric matrices. Addition, multiplication and scalar multiplication of matrices, simple properties of addition, multiplication and scalar multiplication. Non-commutativity of multiplication of matrices and existence of non-zero matrices whose product is the zero matrix (restrict to square matrices of order 2). Concept of elementary row and column operations. Invertible matrices and proof of the uniqueness of inverse, if it exists; (Here all matrices will have real entries).

2. Determinants:

Determinant of a square matrix (up to 3×3 matrices), properties of determinants, minors, cofactors and applications of determinants in finding the area of a triangle. Adjoint and inverse of a square matrix. Consistency, inconsistency and number of solutions of system of linear equations by examples, solving system of

linear equations in two or three variables (having unique solution) using inverse of a matrix.

CALCULUS

1. Continuity and Differentiability:

Continuity and differentiability, derivative of composite functions, chain rule, derivatives of inverse trigonometric functions, derivative of implicit function. Concept of exponential and logarithmic functions and their derivative. Logarithmic differentiation. Derivative of functions expressed in parametric forms. Second order derivatives. Rolle's and Lagrange's Mean Value Theorems (without proof) and their geometric interpretations.

2. Applications of Derivatives:

Applications of derivatives: rate of change, increasing/decreasing functions, tangents & normals, approximation, maxima and minima (first derivative test motivated geometrically and second derivative test given as a provable tool). Simple problems (that illustrate basic principles and understanding of the subject as well as real-life situations).

3. Integrals:

Integration as inverse process of differentiation. Integration of a variety of functions by substitution, by partial fractions and by parts, only simple integrals of the type

$$\int \frac{dx}{x^2 \pm a^2}, \int \frac{dx}{\sqrt{x^2 \pm a^2}}, \int \frac{dx}{\sqrt{a^2 - x^2}}, \int \frac{dx}{ax^2 + bx + c}, \int \frac{dx}{ax^2 + bx + c},$$

$$\int \frac{(px+q)}{ax^2+bx+c} dx, \int \frac{(px+q)}{\sqrt{ax^2+bx+c}} dx, \int \sqrt{a^2 \pm x^2} dx \text{ and } \int \sqrt{x^2 - a^2} dx$$

to be evaluated.

Definite integrals as a limit of a sum.

Fundamental Theorem of Calculus (without proof). Basic properties of definite integrals and evaluation of definite integrals

4. **Applications of the Integrals:**
Applications in finding the area under simple curves, especially lines, arcs of circles/ parabolas/ellipses (in standard form only), area between the two above said curves (the region should be clearly identifiable).

5. **Differential Equations:**
Definition, order and degree, general and particular solutions of a differential equation. Formation of differential equation whose general solution is given. Solution of differential equations by method of separation of variables, homogeneous differential equations of first order and first degree. Solutions of linear differential equation of the type :

$$\frac{dy}{dx} + p(x)y = q(x), \text{ where } p(x) \text{ and } q(x) \text{ are functions of } x.$$

LINEAR PROGRAMMING

1. **Linear Programming:**
Introduction, definition of related terminology such as constraints, objective function, optimization, different types of linear programming (L.P.) problems, mathematical formulation of L.P. problems, graphical method of solution for problems in two variables, feasible and infeasible regions, feasible and infeasible solutions, optimal feasible solutions (up to three non-trivial constraints).

PROBABILITY

1. **Probability:**
Multiplication theorem on probability. Conditional probability, independent events, total probability, Baye's theorem. Random variable and its probability distribution, mean and variance of haphazard variable. Repeated independent (Bernoulli) trials and Binomial distribution.

For Science stream students

1. **Vectors:**
Vectors and scalars, magnitude and direction of a vector. Direction cosines/ ratios of vectors. Types of vectors (equal, unit, zero, parallel and collinear vectors), position vector of a point, negative of a vector, components of a vector, addition of vectors, multiplication of a vector by a scalar, position vector of a point dividing a line segment in a given ratio. Scalar (dot) product of vectors, projection of a vector on a line. Vector (cross) product of vectors.
2. **Three - dimensional Geometry:**
Direction cosines/ratios of a line joining two points. Cartesian and vector equation of a line, coplanar and skew lines, shortest distance between two lines. Cartesian and vector equation of a plane. Angle between (i) two lines, (ii) two planes, (iii) a line and a plane. Distance of a point from a plane.

For Non - Science stream students

Partnership

Basic definitions, sharing of profits, partner's salaries and interest on Capital, Profit sharing on Admission of a New Partner of Retirement of an existing partner.

Bill of Exchange

Bill of Exchange, True Discount, Banker's Discount and Banker's Gain.

Linear Programming

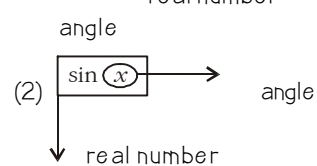
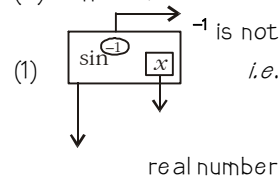
Linear Programming Problems, Different Areas of Applications of Linear Programming Problems, Basic Concepts of Linear Programming Problems, Mathematical Formulation of a Linear Programming Problem, Advantages of Linear Programming Problems, Limitations of Linear Programming, The Graphical Method of Solving and LPP, Some Exceptional Cases.

Mastering Mathematics Skill

Step 1 : Solid foundation of basics and fundamentals

(A) Difference between $\sin^{-1} x$ and $\sin x$.

(1) $\sin^{-1}$ is not an algebraic index.
i.e. $\sin^{-1} x \neq (\sin x)^{-1}$.



Step 2 : Analyse the problem (verbally)

To find the value of

$$\sin \frac{\pi}{14} \sin \frac{3\pi}{14} \sin \frac{5\pi}{14} \sin \frac{7\pi}{14} \sin \frac{9\pi}{14} \sin \frac{11\pi}{14} \sin \frac{13\pi}{14}.$$

$\therefore$ Angles in the increasing order

$$\frac{13\pi}{14} > \frac{11\pi}{14} > \frac{9\pi}{14} > \frac{7\pi}{14} > \frac{5\pi}{14} > \frac{3\pi}{14} > \frac{\pi}{14}$$

(Reduce the larger angles into smaller ones already existing in the expression.)

$$\begin{cases} \sin \frac{13\pi}{14} = \sin \left(\pi - \frac{\pi}{14} \right) = \sin \frac{\pi}{14} \\ \sin \frac{11\pi}{14} = \sin \left(\pi - \frac{3\pi}{14} \right) = \sin \frac{3\pi}{14} \\ \sin \frac{9\pi}{14} = \sin \left(\pi - \frac{5\pi}{14} \right) = \sin \frac{5\pi}{14} \\ \sin \frac{7\pi}{14} = \sin \frac{\pi}{2} = 1 \end{cases}$$

$\therefore$ Given expression becomes

$$\left(\sin^2 \frac{\pi}{14} \sin^2 \frac{3\pi}{14} \sin^2 \frac{5\pi}{14} \right) \text{ or } \left(\sin \frac{\pi}{14} \sin \frac{3\pi}{14} \sin \frac{5\pi}{14} \right)^2$$

[Again reduce the larger angles
Use $\cos \left(\frac{\pi}{2} - \theta \right) = \sin \theta$]

$$\left[\cos \left(\frac{\pi}{2} - \frac{\pi}{14} \right) \cos \left(\frac{\pi}{2} - \frac{3\pi}{14} \right) \cos \left(\frac{\pi}{2} - \frac{5\pi}{14} \right) \right]^2$$

$$\left[\cos \frac{6\pi}{14} \cos \frac{4\pi}{14} \cos \frac{2\pi}{14} \right]^2$$

$$\left[\cos \frac{\pi}{7} \cos \frac{2\pi}{7} \cos \frac{3\pi}{7} \right]^2 \quad [\text{Use } \cos(\pi - \theta) = -\cos \theta]$$

[Use $\sin 2\theta = 2 \sin \theta \cos \theta$]

$$\left[+\cos \frac{\pi}{7} \cos \frac{2\pi}{7} \cos \left(\pi - \frac{4\pi}{7} \right) \right]^2$$

[Multiply and divide inside the bracket by $2 \sin \frac{\pi}{7}$. i.e. by $2 \sin$ (smallest angle)]

$$\left[-\cos \frac{\pi}{7} \cos \frac{2\pi}{7} \cos \frac{4\pi}{7} \right]^2$$

$$\therefore \left[\frac{2 \sin \frac{\pi}{7} \cos \frac{\pi}{7} \cos \frac{2\pi}{7} \cos \frac{4\pi}{7}}{2 \sin \frac{\pi}{7}} \right]^2$$

[Again use the formula $\sin 2\theta = 2 \sin \theta \cos \theta$]

$$\therefore \left[\frac{\sin \frac{2\pi}{7} \cos \frac{2\pi}{7} \cos \frac{4\pi}{7}}{2 \sin \frac{\pi}{7}} \right]^2$$

[Using same formulae]

$$\left[\frac{\sin \frac{4\pi}{7} \cos \frac{4\pi}{7}}{4 \sin \frac{\pi}{7}} \right]^2$$

[Use $\sin(\pi + \theta) = -\sin \theta$]

$$\left[\frac{\sin \frac{8\pi}{7}}{8 \sin \frac{\pi}{7}} \right]^2$$

$$\left[\frac{\sin\left(\pi + \frac{\pi}{7}\right)}{8 \sin \frac{\pi}{7}} \right]^2 \rightarrow \left[\frac{\left(-\sin \frac{\pi}{7}\right)}{\left(8 \sin \frac{\pi}{7}\right)} \right]^2 = \left(\frac{1}{8}\right)^2 = \frac{1}{64}.$$

Step 3 : Use of memories

Memory Frame For Principal Values

Principal value			
		$x > 0$	$x < 0$
		Quadrants	Quadrants
$\sin^{-1} x$	$\rightarrow$	1	4
$\cos^{-1} x$	$\rightarrow$	1	2
$\tan^{-1} x$	$\rightarrow$	1	4
$\cot^{-1} x$	$\rightarrow$	1	2
$\sec^{-1} x$	$\rightarrow$	1	2
$\operatorname{Cosec}^{-1} x$	$\rightarrow$	1	4

Step 4 : Analyse the difference between the statements of problems

1. Find the number of tangents which can be drawn from the point (1, 2) to the circle $x^2 + y^2 = 5$.

Soln. Equation of circle : $x^2 + y^2 = 5$... (1)

Given point is (1, 2).

Substitute the point in equation (1), we get $1^2 + 2^2 = 5$. $\therefore$ Point lies on the circle.

$\therefore$ we can not draw any tangent from (1, 2) to the circle, since point itself lies on the circle.

2. Find the number of tangents which can pass through (1, 2), to the circle $x^2 + y^2 = 5$.

Soln. Equation of circle : $x^2 + y^2 = 5$... (1)

Given point is (1, 2)

Substitutes the point in the equation (1) $1^2 + 2^2 = 5$. $\therefore$ Point lies on the circle.

Number of Tangent's which can pass through (1, 2) is only one.

Step 5 : Tips to solve a particular problem

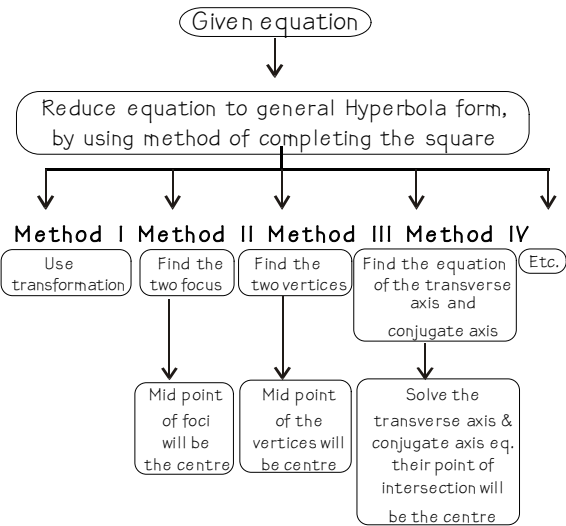
To find the centre of the Hyperbola $9x^2 - 16y^2 - 18x - 32y - 151 = 0$.

Soln. We solve the problem by knowing the various

facts involved, firstly let us define centre;

- (i) Mid point of the two focus (foci)
- (ii) Mid point of the two vertex (vertices)
- (iii) Point of intersection of the transverse and conjugate axis
- (iv) Bisects every chord of the hyperbola passes through it.

After knowing the various facts, the student has the following procedure to find centre.



But all the above methods require more time. We shall provide you a simplest method how to find the centre of Hyperbola.

Simplest Method

Given equation :

$$9x^2 - 16y^2 - 18x - 32y - 151 = 0 \quad \dots (i)$$

Firstly differentiate equation (i) w.r.t. x keeping y as constant (this is called partial differentiation w.r.t. x), we get, $18x - 18 = 0$... (ii)

Secondly differentiate equation (i) w.r.t. y keeping x as constant. (this is called partial differentiation w.r.t. y), we get, $-32y - 32 = 0$... (iii)

Solving, we get, $x = 1$ and $y = -1$
 $\therefore$ Required centre being (1, -1).

Your assignment:

You are required to apply the above mentioned steps to solve the following problems.

Right Parabola $y^2 = 4ax$.

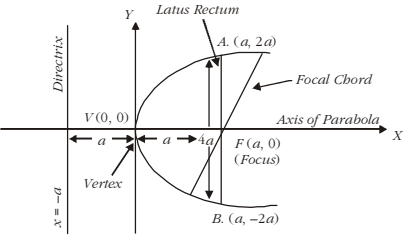
Points to be discussed:

- (i) Shape of curve
- (ii) Tangent at vertex V

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Activity work sheet

(iii) Symmetry of curve (iv) Extension of curve



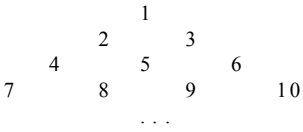
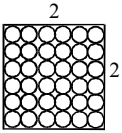
Questions :

- Is every focal chord a latus rectum ?
- Is every latus rectum a focal chord ?
- If the axis of parabola is along the +ve direction of x - axis and focus is at a distance of 2 units from origin. Then what is the equation of the directrix?

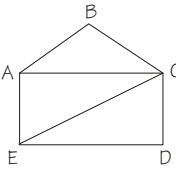
- What is the equation of the parabola with vertex at the origin and having its axis along the x - axis and passing through $(1, -4)$?
- If $(2, 0)$ is the vertex and y - axis the directrix of parabola, then what is the focus ?
- If focus of the parabola is at $(3, 3)$ and its directrix is $3x - 4y = 2$. What is latus rectum ?
- What does the following equations, represents?
(A) $y^2 - x - 2y + 2 = 0$
(B) $25x^2 + 9y^2 - 150x - 90y + 225 = 0$
(C) $3x^2 + 10xy + 8y^2 + 14x + 22y + 7 = 0$
- If $P(-3, 2)$ is one end of the focal chord PQ of the parabola $y^2 + 4x + 4y = 0$. Then what is the slope of normal at Q ?
- An equilateral triangle is inscribed in the parabola $y^2 = 4ax$ whose one vertex is at the vertex of the parabola. What is the length of the side of triangle?

Brain Teasers

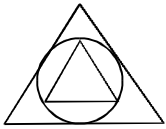
- Four letters are randomly chosen from the word MATHEMATICS. What is the probability that the four letters can be used to spell MATH?
- A square oven tray 2 meters on a side filled with n^2 circular cherry pies. What is the total area of the pie crusts on top of the pies?
- Three equilateral triangles with height 9 are arranged to form an isosceles trapezoid. Where is the centroid of the trapezoid?
- The positive integers are written in a triangular array as shown. In what row will the number 1000 be?



- $ABCE$ is an isosceles trapezoid and $ACDE$ is a rectangle. $AB = 10$ and $EC = 20$. Find AE .



- A circle is circumscribed about one equilateral triangle and is inscribed in another equilateral triangle. Find the ratio of the areas of the triangles.



Discover The Human Calculator in You

In his Oscar - winning role in the movie *Rain Man*, Dustin Hoffman plays an idiot savant who can perform complex calculations instantly in his head. Like the “*Rain Man*”, various people have displayed their outstanding capacity for mental arithmetic on TV, and many others have written books teaching these powerful techniques. However, hardly any have ever ventured to justify the validity of these algorithms, as the mathematics involved is surprisingly elementary. In this article, we detail some of the famous tricks that the “*human calculators*” have used over the years, explain why these methods work, and you will see that, with a little practice, you too can be a human calculator.

Trick 1 : Squaring two - digit numbers ending in 5.

To square any two - digit number that ends in 5, add one to the first digit and multiply that sum by the first digit. This will be the first two digits of the answer. The last two digits will always be 25.

For example, $85^2 = 7225$ since $8 \times (8 + 1) = 8 \times 9 = 72$, and likewise, $25^2 = 625$ since $2 \times 3 = 6$. We can extend this to larger numbers, for example $195^2 = 38025$, since $19 \times 20 = 380$.

If you are wondering why this method works, a little algebra will quickly convince you : $(10A + 5)^2 = 100A^2 + 100A + 25 = 100A(A + 1) + 25$.

Thus, the first two digits will be $A(A + 1)$, and the last two digits will be 25.

Let us take this idea one step further. Let us multiply pairs of two - digit numbers whose tens digits are the same, and whose units digits sum to ten. For example, $37 \times 33 = 1221$, $36 \times 34 = 1224$, and $98 \times 92 = 9016$.

Do you see the pattern? Like in the case above, the first two digits of the answer are determined in the same way. But what about the last two digits? Do you see how they are obtained? If so, use a little algebra and convince yourself that it always works.

Trick 2 : Squaring any two - digit number.

Take any two - digit number n . Now we know that

$$n^2 = (n^2 - d^2) + d^2 = (n - d)(n + d) + d^2$$

So let us try to find a value of d so that the product $(n - d)(n + d)$ can be easily calculated. Consider the multiple of 10 that is closest to n , and let the difference between the number and this multiple of 10 be d . For example, if we take $n = 87$, then the multiple of 10 that is closest to 87 is 90, and since $90 - 87 = 3$, we have $d = 3$. Similarly, if $n = 94$, we have $d = 4$.

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Activity work sheet

If we perform this calculation for any integer n , then one of $n - d$ or $n + d$ will be a multiple of 10, and the calculation becomes significantly easier. The following examples illustrate this technique :

$87^2 = (87 + 3) (87 - 3) + 3^2 = 90 \times 84 + 9 = 7569$.

$29^2 = (29 + 1) (29 - 1) + 1^2 = 30 \times 28 + 1 = 841$

$96^2 = (96 + 4) (96 - 4) + 4^2 = 100 \times 92 + 16 = 9216$.

Use this technique to compute the following :

$37^2, 52^2, 1999^2$.

Trick 3 : The calender trick.

One of the more interesting demonstrations performed by “*mathemagicians*” is the calender trick. Namely, an audience member calls out her birthday, or some historical date, and the human calculator is able to tell her what day of the week that event took place. The first thing to do is to memorize the following table.

Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
1	4	4	0	2	5	0	3	6	1	4	6

It appears challenging to remember this, but there is an interesting pattern here. Reading the row of numbers from left to right in threes, we have 144, 025, 036 and 146. Notice that 144, 025 and 036 are perfect squares, and the last number is just 2 more than the first perfect square, 144. This should make the memorization easier.

Let Y be the last two digits of the year in question. Let D be the day we are searching for, and let M be the integer that corresponds to the month in the above table. Thus, if we are searching for July 25th, 1978, Y would be 78, D would be 25, and M would be 0, since July corresponds to 0. Compute the value of

$$Y + \left\lfloor \frac{Y}{4} \right\rfloor + D + M,$$

and divide that sum by 7. Whatever remainder you get corresponds to the day of the week you are seeking, namely 0 is Saturday, 1 is Sunday, 2 is Monday, 3 is Tuesday, 4 is Wednesday, 5 is Thursday and 6 is Friday.

A small note to remember. If the year is a leap year, and the month is January or February, you must subtract 1 from the total. This is due to the fact that the extra day in a leap year occurs on February 29th, and so if the day you are searching is before that, then the formula is off by one day.

Let us look at a historical date in the 20th century. The famous stock market crash of 1929 occurred on October 29th, so let us use our formula to determine what day of the week “Black Tuesday” occurred.

Activity work sheet

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We have $Y = 29$, $M = 1$ and $D = 29$. Hence $\left\lfloor \frac{29}{4} \right\rfloor = 7$, and our sum is $29 + 7 + 29 + 1 = 66$. Dividing this number by 7, we find that the remainder is 3. We conclude that October 29th, 1929 was indeed a Tuesday.

Unfortunately, this formula only works with dates in the 1900's because in the Gregorian calendar, not all years that are divisible by 4 are leap years. For example, 1800 and 1900 are not leap years, but 2000 is. And thus, we must alter our formula to compensate for this. To calculate dates in the 1800's, use the same formula, but go forward two days in the week. To calculate dates in the 2000's, go one day back. If we use our formula, we find that January 1st, 2000 is a Sunday (remember, 2000 is a leap year!). Go back one day, because it really is a Saturday.

In the past, it was believed that a year had precisely 365.25 days, and so we compensated for the extra quarter day by adding February 29th to our calendar once every four years. Unfortunately, a year has 365.2422 days, so we cannot add an extra day exactly once every years. It would be nice however if we could, for then this formula would always hold.

Using this method, determine what day of the week you were born on.

Trick 4 : Extracting Cube Roots.

We now detail the method for determining the cube roots of all perfect cubes under one billion.

First, you must first learn the cubes of the integers 0 through 9.

n	0	1	2	3	4	5	6	7	8	9
n^3	0	1	8	27	64	125	216	343	512	729

Let us first find the cube root of numbers that are below one million. Hence, the cube root will be at most 99. Say we want to find the cube root of 314, 432. We separate the number into two parts, separated by the comma. Thus, 314 is the first part, and 432 is the second part. The desired cube root has two digits. We will use the first part to get the first digit, and we will use the second part to get the second digit.

Take the first part and determine where it lies in the table of cubes. In our example, 314 lies between 216 and 343. Thus $216,000 < 314,432 < 343,000$, which implies that the desired cube root lies between 60 and 70, since $60^3 = 216,000$ and $70^3 = 343,000$. Hence, it follows that the first digit of our cube root must be 6.

Now we determine the second digit. If we look at the table of cubes, notice that each cube ends in a different digits. So if a certain cube ends in 2, we know that its cube root must end in 8, because 8 is the only digit whose cube ends in 2. Since 432 ends in 2, the second digit of the cube root must be 8. Thus, the desired cube root is 68.

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